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Elapsed Time: Why Is It So Difficult to Teach?

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Based on Piaget's theory of logico-mathematical knowledge, 126 students in grades 2–5 were asked 6 questions about elapsed time. The main reason found for difficulty with elapsed time is children's inability to coordinate hierarchical units (hours and minutes). For example, many students answered that the duration between 8:30 and 11:00 was 3 hours 30 minutes (because from 8:00 to 11:00 is 3 hours, and 30 more minutes is 3 hours 30 minutes). Coordination was found to begin among logico-mathematically advanced students, through reflective (constructive) abstraction from within. The educational implications drawn are that students must be encouraged to think about durations in daily living and to do their own thinking rather than being taught procedures for producing correct answers to elapsed-time questions.

Key words: Constructivism; Curriculum; Elementary K–8; Piaget

When we ask teachers in graduate courses on primary mathematics what is especially difficult for them to teach, the answer they often give is *elapsed time*. If we ask what makes elapsed time so difficult to teach, the teachers become unsure. We searched the literature to find out what is known about this difficulty but found no pertinent research.

Although we did not find any reference to children's difficulty with elapsed time, publications about children's concept of *time* are numerous. For example, Friedman (1982) studied children's ideas about conventional time (e.g., days, weeks, and months). Fivush and Mandler (1985) examined children's ability to sequence events forward and backward. French (1989) studied how students sequence events using the word *before* or *after*. She found that when pictures of three sequential events (A, B, and C) were presented, children had a strong preference for saying that B took place *after* A rather than saying that B took place *before* C.

Many studies have been inspired by Piaget's research (1969, 1970) but not by his theory about logico-mathematical knowledge. For example, Murray (1969) investigated the conservation aspects of durations by showing a drawing of two pennies labeled A and B, and telling the respondent that the pennies were made at the same time. Coin A looked older because it was dropped in an acid, and coin B looked new. The children were asked if A was as old as B, or whether one of them was older. Siegler and Richards (1979) examined children's judgments of duration,

speed, and distance. Their tasks were designed to find out on what children base their judgments: the end point, the end time, the beginning point, the beginning time, duration, speed, and/or distance. Levin (1982), too, studied children's judgments of duration, speed, distance, and end points without awareness of the nature of logico-mathematical knowledge.

Piaget (1971, 1951) made a fundamental distinction among three kinds of knowledge according to their ultimate sources: physical, logico-mathematical, and social-conventional knowledge. In his conceptualization, temporal relationships are part of logico-mathematical knowledge.

Physical knowledge is knowledge of objects in the external world. The knowledge that a penny is made of metal is an example of physical knowledge. Knowing that a watch is made with metal and glass and that glass breaks when it is dropped is also an example of physical knowledge. By contrast, the fact that there are 60 minutes in an hour and 24 hours in a day is *social-conventional knowledge*. Having to use one's right hand for handshakes is likewise an example of social knowledge, for which the ultimate source is conventions that people establish. Social-conventional and physical knowledge are similar in that their ultimate sources are outside the individual.

Logico-mathematical knowledge is different in that it consists of mental relationships that each individual creates in his or her mind. For example, if one is presented with a man's watch and a woman's watch, one can say that they are *different*. One also can say that the two watches are *similar*, and it is just as true to say that the two watches are similar as it is to say that they are different. The difference and similarity are *mental relationships* that each person makes (constructs) in his or her mind and are examples of logico-mathematical knowledge that each individual creates starting in infancy (Piaget, 1954). *Bigger than, later than*, and *two* are also examples of logico-mathematical knowledge.

Piaget studied five kinds of logico-mathematical relationships: classificatory, seriational (Inhelder & Piaget, 1964), numerical (Piaget & Szeminska, 1952), spatial (Piaget, Inhelder, & Szeminska, 1960), and temporal (Piaget, 1969) relationships. In *The Child's Conception of Time*, Piaget (1969) made an important distinction between *intuitive* (preoperational) time and *operational* (logico-mathematical) time. Children begin by thinking intuitively about time, and construct operational time between 7 and 8 years of age, according to Piaget, during the concrete-operational period. Intuitive time is based on what children experience and observe empirically, as can be seen in the following example from an interview Piaget conducted with a 4-year-old:

> ROM . . . has a small sister called Erica: How old is she? *Don't know*. Is she a baby? *No, she can walk*. Who is the older of you two? *Me*. Why? *Because I'm the bigger one*. Who will be older when she starts going to school? *Don't know*. When you are grown up, will one of you be older than the other? *Yes*. Which one? *Don't know*. (p. 203)

ROM based her judgments of age on size, which was observable. She could tell that she was older than her sister because she was bigger, but she could not deduce

logico-mathematically that she would always be older than Erica. She did not know who would be older when she went to school or became a grownup, because she did not know who would be bigger.

When ROM becomes 7 or 8, according to Piaget, she is likely to "group" (coordinate) the temporal relationships she constructed before, such as durations and the order of events, and construct an operational organization of time. When time becomes operational, children become able to fit objects and events within a temporal framework and deduce logico-mathematically that if ROM is older than Erica now, ROM will always be older than Erica.

Kamii and Russell (2010) replicated one of Piaget's (1969) tasks involving a pear tree that was planted 1 year later than an apple tree but grew to be larger than the apple tree. Based on results from this task, they concluded that children generally construct operational time by third grade.

The questions asked in both Piaget's and Kamii and Russell's interviews involved qualitative quantification. Qualitative quantification involves only three categories such as *the same age, older than*, and *younger than*. Numerical quantification is more precise and develops later. A question asking for numerical quantification would have been, "By how many years will you be older than Erica when you start going to school?"

Elapsed time involves numerical quantification, and Long and Kamii (2001) investigated children's ability to measure time with unit iteration within a Piagetian framework. One of their findings was that children generally become able to iterate units of time late in sixth grade. Because questions about elapsed time always involve units, we can expect elapsed time to remain difficult until sixth grade.

In questions about elapsed time, units occur at two hierarchical levels. For example, to determine the duration between 8:30 and 11:00, children have to know that 8 is one kind of number (at a higher hierarchical level) and that 30 is another kind of number (at a lower level), as illustrated in Figure 1(a). The knowledge that there are 60 minutes in an hour is social-conventional knowledge, but the thinking required to coordinate hours and minutes simultaneously is logico-mathematical knowledge. This hierarchical thinking has the same structure as the thinking necessary to deal with 10s and 1s (see Figure 1(b)). This hierarchical structure is also required for multiplicative thinking, as we will explain later.

Children have long been known to have trouble with the coordination of 10s and 1s (e.g., Fuson, 1990, 1992). For example, Labinowicz (1985) individually asked 29 third-grade students to count the blocks shown on the left side of Figure 2a. He then took the cover off to reveal all the blocks, as shown on the right side of Figure 2a, and asked each child to "count how many cubes there are on the board altogether (p. 255)." Figure 2b shows that Zoya counted either by 10s (incorrectly) or by 1s (incorrectly). Apparently, she could not shift from counting by 10s to counting by 1s, or vice versa, because she could think at only one hierarchical level at a time.

Ross (1985) asked children to put 10 beans into each cup as illustrated in Figure 3a and asked them how many beans there were altogether. Twenty-seven percent of the second-grade and third-grade students counted them as shown in Figure 3b and announced that there were 80. (Some second-grade students were at an even lower

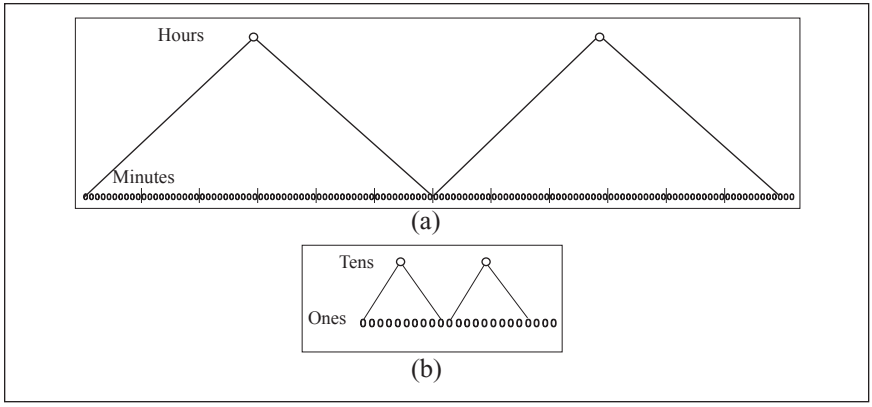


Figure 1. The hierarchical thinking involved in coordinating hours and minutes (a) and 10s and 1s (b).

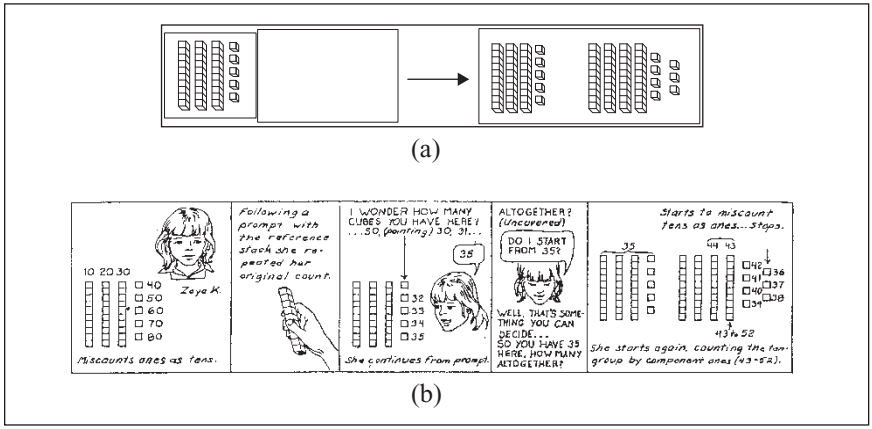


Figure 2. A counting board (a) and a Zoya's response to a counting board task (b) from *Learning From Children: New Beginnings for Teaching Numerical Thinking: A Piagetian Approach* by E. Labinowicz, 1985, p. 255. Copyright 1985 by Addison-Wesley Publishing Company.

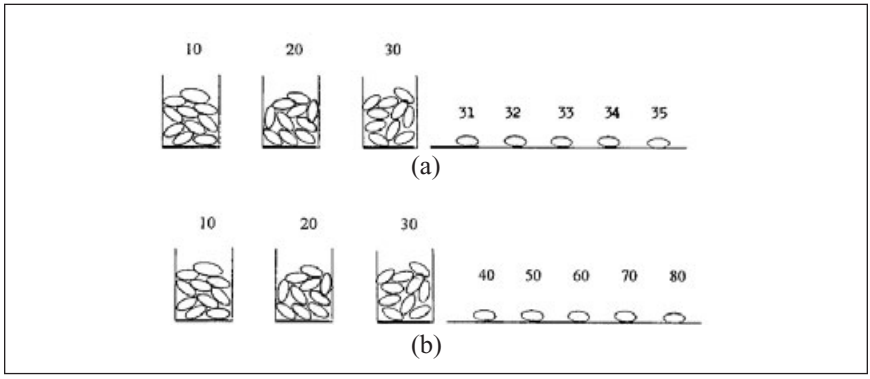


Figure 3. Two ways of counting by 10s.

level.) Adults and logico-mathematically more advanced children can shift to 1s (see Figure 3a) because they can think about the 1s while counting by 10s. Otherwise, they would continue to count by 10s as illustrated in Figure 3b.

More recently, Chandler and Kamii (2009) found that 25% of the second graders in their sample insisted that a dime could not be used to pay for 6 cents worth of candy in a store game. The children all knew verbally that a dime was worth 10 pennies, and that 10 pennies was "too much" for a 6-cent purchase. Yet, for them, a dime was one thing, and 10 pennies was something else. Adding a few pennies to a dime was easy for these second graders because addition allowed them to keep "one 10" intact, as shown in Figure 4a. However, subtracting pennies from a dime was impossible because a dime was a unit that could not be separated into 10 lower order units (see Figure 4b).

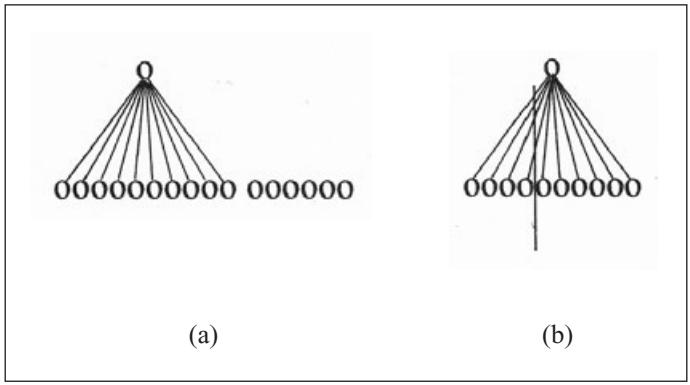


Figure 4. Adding 6 cents to a dime (a) compared with subtracting 6 cents from a dime (b).

The primary research that explains how children become able to coordinate 10s and 1s is that of Steffe and colleagues (Steffe, 1992; Steffe & Cobb, 1988). By analyzing the ways children count objects to solve problems, Steffe (1992) brought to light how children constructed *iterable composite units*. Composite units are higher order units of two or more lower order units. For example, when adults count by 2s, they are making composite units of 2. When they count by 10s, likewise, adults are making composite units of 10. First-grade students can usually count only by 1s, but they gradually construct higher order composite units and become able to count by 2s, 5s, and 10s. When children have solidly constructed these higher order units, the higher order units are said to have become *explicitly iterable composite units*.

Steffe explained children's construction of 1s as well as of composite units by referring to *reflective* (or *constructive*) *abstraction*. Piaget and colleagues (e.g., Beth & Piaget, 1961/1966) distinguished between *empirical* abstraction and *reflective* (or *constructive*) abstraction. In empirical abstraction, we focus on one or more observable properties of objects (such as color) and ignore the others. When we

sort 10 red and blue circles by color, we engage in empirical abstraction. By contrast, when we *make* (construct) mental relationships such as *different, similar,* and *two* between a red circle and a blue one, we engage in reflective (or construct ive) abstraction. All logico-mathematical relationships, including lower order and higher order units, are constructed by each individual by reflective (constructive) abstraction, through his or her own mental actions (thinking).

More recently, Steffe and Olive (2010) extended their work about the coordination of units to the realm of fractions. For example, when asked to cut up a 1/4 stick so that each piece will be 1/20 of a whole stick, children have to coordinate three levels of units: the levels of 1s, fourths, and twentieths. Only by coordinating the three levels can a child determine that the 1/4 stick must be cut into 5 pieces. Steffe and Olive cited many examples of children's efforts leading to successful and unsuccessful responses that illustrated the difficulty of this coordination. Not surprisingly, in the realm of fractions, too, they explained the construction and coordination of iterable composite units as resulting from reflective abstraction.

Steffe (1992) and Clark and Kamii (1996) have used the concept of reflective abstraction to explain students' construction of multiplicative thinking. Most mathematics educators view multiplication as only a faster way to do repeated addition (see Figure 5a). However, Steffe as well as Clark and Kamii showed that multiplication involves the two hierarchical levels apparent in Figure 5b. It can be noted in this figure that the *4* in *4* × *5* is not the same kind of number as the *5*. The *4* is a higher order number that refers to "4 groups" (of 5 objects). According to Clark and Kamii (1996), children typically begin to understand these higher order numbers by Grade 4, and only 49% of the fifth-grade students they interviewed immediately understood terms such as *two times* and *three times*.

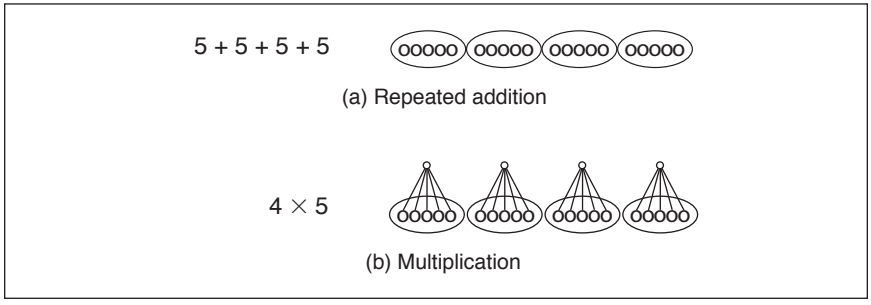


Figure 5. The thinking involved in repeated addition (a) and in multiplication (b).

Reynolds and Wheatley (Reynolds & Wheatley, 1996; Wheatley & Reynolds, 1996) studied fourth-grade students' coordination of numerical and spatial units. By asking "How many 3-by-5 cards are needed to cover a sheet of paper 12 by 30?" (Reynolds & Wheatley, 1996, p. 567), for example, they found out how the children coordinated numerical units (3, 5, 12, and 30) and spatial units (3 by 5 and 12 by 30) at the same time. They concluded that when a student constructs and coordinates

higher order units in the numerical realm, he or she also constructs and coordinates higher order units in the spatial realm, by reflective abstraction.

The literature on how to teach elapsed time suggests that textbooks first show children how to count hours on an analog clock, and then how to count minutes, *separately*. Some textbooks then show how to get elapsed time by using the social-conventional rule of "borrowing" 60 minutes, as illustrated in Figure 6a.

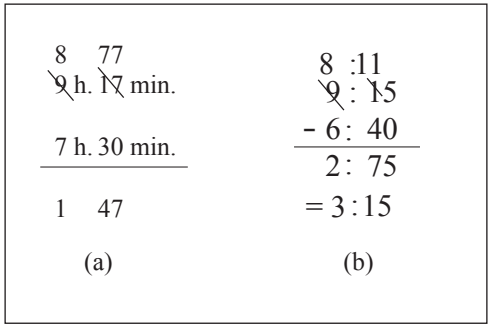


Figure 6. "Borrowing" shown in a textbook (a) and carried out by a student (b).

Countless articles have been published giving advice to teachers on how to teach elapsed time. For example, Moone and de Groot (2005) recommended instructing children to draw empty number lines, with large "jumps" for hours and small "jumps" for minutes. According to the authors, this activity was "intended as a scaffold toward" an algorithm such as the one in Figure 6a involving regrouping.

To summarize what is known from the research literature about children's logico-mathematical development of the concept of elapsed time, it is known first of all that children become able to deduce elapsed time qualitatively by Grade 3 (Kamii & Russell, 2010). It is also known that children become able to measure time with unit iteration in Grade 6 (Long & Kamii, 2001). Because elapsed time involves hierarchical units of time, we can expect it to remain difficult until at least Grade 6.

In response to the total absence of research attempting to explain why elapsed time is so difficult for children, we decided to ask children questions that are similar to those found in textbooks. We decided to begin with an easy question involving only whole hours and to add increasingly difficult minutes, as do textbooks. In other words, the research reported in this article was exploratory. We thought that a good way of beginning to explain the difficulty of elapsed time was to determine what kinds of questions are more difficult than others.

METHOD

In individual interviews, we asked a total of 126 students in Grades 2–5 six questions about elapsed time. The students (29 in second grade, 29 in third grade, 33 in fourth grade, and 35 in fifth grade) were members of two classes at each grade level

whose parents gave permission for the interview. The school was in an upper-middle-class, suburban neighborhood near a large city in the South. The racial composition of the school was 69% Caucasian, 18% African American, 9% Asian, and 4% Hispanic. The interviews took place in January and February.

Grades 2–5 were selected because textbooks typically show how to teach elapsed time in third and fourth grade, and second graders would show a "floor" with respect to elapsed time. In fifth grade, elapsed time typically no longer appears in textbooks, presumably because children are believed to have mastered it.

The second-grade teachers in the school did not teach elapsed time because previous experience convinced them that such attempts were not fruitful. In grades 3–5, by contrast, the teachers taught elapsed time in the context of daily living. For example, they sometimes asked the class how much time they had before lunch, recess, and/or dismissal time, and whether the amount of time available was enough to play a mathematics game.

The six interview questions were based on findings from a pilot study. In the pilot study, we asked the 10 questions that appear in Table 1.

As illustrated in Table 2, Question 1 was an easy one by third grade, with 78% to 92% of students answering correctly. We decided to retain this question for the present study because it served as a "floor."

Questions 2 through 5—involving 30 minutes—yielded percentages of correct answers that were similar but much lower than those for Question 1. Whether the 30 minutes came at the beginning or the end of a duration, the percentages were similar, especially in Grades 3 and 4. We therefore decided to ask only one question involving 30 minutes in the present study. This question became Question 2 in the present study, with the 30 minutes appearing at the beginning of the interval.

Questions 6 and 7 involved more precise minutes that made them more difficult than Questions 2–5, especially in Grades 3 and 4. We decided to retain these questions for the present study but changed their order. Question 3 of the present study thus involved only quarter-hours, and Question 4 of the present study involved more precise minutes.

Because Questions 8 and 9 of the pilot study yielded percentages that were similar to Questions 6 and 7, we decided to omit them in the present study to avoid useless repetition. Question 10 of the pilot study was much too difficult, even for fifth-grade students. We therefore decided to replace it with an easier question that became Question 5 of the present study.

The present study thus used the six questions in Table 3. We hypothesized that Questions 2–6, which involved the coordination of hours and minutes, would be much more difficult than Question 1, which dealt with whole hours.

In each interview, the child and interviewer each had a copy of the six questions, and the child was asked to read each question aloud before answering it. The interviewer wrote the answer the child gave to each question and often asked how he or she figured it out. The explanation and occasional self-corrections were recorded as well as responses to other probes. When the child sat for a long time without responding, he or she was asked if a pencil might help. The answer was usually "No," and the child was asked if a picture of clocks might be useful.

Table 1
Questions Asked in the Pilot Study

Question	Purpose
1. To go "trick or treat" on Halloween night, I left the house at 7:00 and came home at 9:00. How long was I gone?	To determine how the child dealt with whole hours
2. The swim meet will start at 10 o'clock and end at 12:30. How long will it take?	To determine how the child dealt with 30 minutes at the end of the interval
3. My father dropped me off at the movie theater at 11 o'clock, and my mother picked me up at 1:30. How long was I in the movie theater?	To determine how the child dealt with 30 minutes at the end of an interval that spans 12:00
4. I have a friend whose parents say it takes 3 hours to drive to Nashville. If they get going at 7:30 in the morning, what time will they get to Nashville?	To determine how the child dealt with 30 minutes at the beginning of the interval
5. I have another friend whose parents like to get going later in the day. If they get going at 10:30 in the morning and expect to take 4 hours, what time will they be in Nashville?	To determine how the child dealt with 30 minutes at the beginning of an interval that spans 12:00
6. I started working on my homework at 8:15 and finished at 9:10. How long did it take?	To determine how the child dealt with an interval less than an hour and with minutes more precise than quarter-hours at one end point
7. The swimming pool is open from 9:15 in the morning to 11:30 every morning during the summer vacation. How long is it open every day?	To determine how the child dealt with 15 minutes and multiples of 15 minutes at both end points of the interval
8. We had company last night and started to eat at 6:10. When we finished, it was 7:45. How long did we sit at the dinner table?	To determine how the child dealt with minutes more precise than quarter-hours at one end point
9. I started working on a test at 9:05 and finished at 11:00. How long did I work on this test?	To determine how the child dealt with minutes more precise than quarter-hours at one end point
10. My mother volunteers at the animal shelter every Tuesday from 9:30 in the morning to 2:20 in the afternoon. How much time does she volunteer on Tuesdays?	To determine how the child dealt with many hours and precise minutes that span 12:00

Table 2
Percent of Children in the Pilot Study at Each of Grades 2–5 That Gave Correct Answers

Questions	Grade level			
	2 $n = 33$	3 $n = 32$	4 $n = 37$	5 $n = 38$
1. (7:00–9:00)	39	78	87	92
2. (10:00–12:30)	39	63	59	79
3. (11:00–1:30)	30	56	59	66
4. (3 hrs. after 7:30)	15	56	54	58
5. (4 hrs. after 10:30)	24	56	51	71
6. (8:15–9:10)	12	31	28	58
7. (9:15 to 11:30)	15	28	30	55
8. (6:10–7:45)	9	31	32	47
9. (9:05–11:00)	12	25	22	58
10. (9:30 am to 2:20 pm)	3	19	14	34

RESULTS

Data analysis began by counting the number of correct answers the children gave to each question at each grade level. These frequencies were changed to the percentages in Table 4.

Question 1 (*7:00–9:00*)

This question was easy, even for second graders. Sixty-nine percent of the students in Grade 2 gave the correct answer of 2 hours, and the percentages quickly rose to 93%, 97%, and 97%, for Grades 3, 4, and 5, respectively.

The most frequently given incorrect answer in second grade was 1 hour. This answer was given by 28% of the second graders, who explained that there was only one number between 7 and 9. For them, 7:00 and 9:00 may have been mere numbers rather than two end points of a time interval.

Question 2 (*8:30–11:00*)

The percentages of students giving the correct answer to this question (see Table 4) were dramatically lower than those for Question 1. Only 28%, 45%, 61%, and 60%, respectively, of those in Grades 2, 3, 4, and 5 gave the correct answer. According to the sign test for paired samples (Siegel & Castellan, 1988), the decrease in percentage correct from Question 1 to Question 2 was statistically significant ($p < .001$) at each grade level. These decreases revealed that even the coordination of hours and half-hours was difficult for children in Grades 2–5.

Table 3
Six Questions Used in the Present Study

Question	Purpose
1. To go "trick or treat" on Halloween night, I left the house at 7:00 and came home at 9:00. How long was I gone?	To determine how the child dealt with whole hours
2. To drive to Nashville, we left the house at 8:30 in the morning and got there at 11:00. How long did it take to drive to Nashville?	To determine how the child dealt with 30 minutes at the beginning of an interval
3. The swimming pool is open from 9:15 to 11:30 every morning during the summer vacation. How long is it open every day?	To determine how the child dealt with 15 minutes and multiples of 15 minutes
4. I started doing my homework at 8:15 and finished at 9:10 last night. How long did it take?	To determine whether the child could coordinate hours and minutes more precise than quarter-hours
5. I went to Chicago by plane. We left Birmingham at 6:40 and arrived in Chicago at 9:15. How long did the flight take?	To determine whether the child could coordinate hours and minutes and deal with minutes greater than 30. For example, the 40 minutes in 6:40 had to be recognized as 20 minutes before 7:00.
6. When my mother drives, it takes 2 hours and 20 minutes to get to Atlanta. If we want to be in Atlanta by 4:00 in the afternoon, what time should we get started?	To determine whether the child could think about elapsed time in reverse, with nonzero minutes at only one end point of the interval

Table 4
Percent of Children at Each Grade Level That Gave the Correct Answer to Questions 1–6

Questions	Grade level			
	2	3	4	5
	$n = 29$	$n = 29$	$n = 33$	$n = 35$
1. (7:00–9:00)	69	93	97	97
2. (8:30–11:00)	28	45	61	60
3. (9:15 –11:30)	21	34	52	69
4. (8:15–9:10)	21	24	52	40
5. (6:40–9:15)	17	10	30	31
6. (By 4:00)	28	41	52	54

Table 5 lists the most frequently given incorrect answers to Questions 2–5. Note that the most frequent incorrect answer to Question 2 was 3 hours and 30 minutes. The percentages giving this answer were 34, 38, 33, and 31 for Grades 2–5, respectively—about one third at each grade level. This answer was obtained by counting 3 hours from 8:00 to 11:00 and adding 30 minutes to the 3 hours. If these children had reasoned that the duration started 30 minutes after 8:00, they would have subtracted 30 minutes from the 3 hours instead of adding them. Most of the other incorrect answers to Question 2 were distributed as shown in Table 6. This table is presented to indicate that no pattern could be observed from the other incorrect answers given. Likewise, we observed no patterns in the incorrect answers to Questions 3–6.

Question 3 (*9:15–11:30*)

This question was formulated to determine how children dealt with durations beginning and ending with quarter-hours. It was the only one of the six questions that yielded steady growth from Grade 2 to Grade 5 (21%, 34%, 52%, and 69%). About the same percentage of students in Grades 2–5 added the minutes (from 9:00 to 11:00 is 2 hours, and 15 + 30 = 45 minutes).

Table 5
Percent of Children at Each Grade Level That Gave the Most Frequent Incorrect Answers to Questions 2–5

	Grade level			
	2 *n* = 29	3 *n* = 29	4 *n* = 33	5 *n* = 35
Question 2 (8:30–11:00)				
3 hours 30 minutes	34	38	33	31
Question 3 (9:15–11:30)				
2 hours 45 minutes (15 + 30)	17	28	18	17
Question 4 (8:15–9:10)				
1 hour or more	76	62	48	54
1 hour 25 minutes (15 + 10)	24	17	18	14
1 hour 5 minutes (15 – 10)	3	10	21	34
Question 5 (6:40–9:15)				
3 hours 55 minutes (40 + 15)	28	31	21	20
2 or 3 hours and 25 minutes (40 – 15)	3	7	15	23
3 hours 35 minutes (40 – 15)	3	3	6	17

Table 6
Frequencies of Incorrect Answers Other Than the Ones Reported in Table 5 Given by Children at Each Grade Level to Question 2 (*8:30–11:00*)

	Grade level			
	2 *n* = 29	3 *n* = 29	4 *n* = 33	5 *n* = 35
1 1/2 hours	0	1	0	2
2 hours	1	2	0	0
3 hours	6	2	0	0
4 hours	1	1	1	0
4 1/2 hours	1	1	1	1

Question 4 (*8:15–9:10*)

Questions 4 and 5 especially showed the difficulty of coordinating hours and minutes. To answer Question 4 correctly, the child had to coordinate hours and minutes and notice that the duration was less than 1 hour. As illustrated in Table 4, the percentages of students giving the correct answer were low in Grades 2 and 3 (21% and 24%), and a little higher in Grades 4 and 5 (52% and 40%).

In Table 5, it can be noted that half or more of the students in Grades 2–5 (76%, 62% 48%, and 54%) gave an answer of 1 or more hours to Question 4. The answer of 1 hour 25 minutes, given by indiscriminately adding the minutes, decreased slightly from 24% in second grade to 14% in fifth grade, but the answer of 1 hour 5 minutes, given by indiscriminately subtracting the smaller number from the larger number increased from 3% in second grade to 34% in fifth grade.

Question 5 (*6:40–9:15*)

The purpose of this question was to find out whether the child could not only coordinate hours and minutes but also recognize the 40 minutes after 6:00 as 20 minutes before 7:00. The percentages in Table 4 show that Question 5 was the most difficult of all six questions. Only 17%, 10%, 30%, and 31%, respectively, answered correctly in Grades 2–5.

The most frequently given incorrect answer to Question 5 was 3 hours 55 minutes, according to Table 5, obtained by dealing with the hours first (from 6:00 to 9:00 is 3 hours) and mechanically adding the minutes. Other students simply subtracted the smaller number of minutes from the larger number of minutes (40 – 15 = 25 or 40 – 15 = 35). The latter, 40 – 15 = 35, is a common error found among students whose mental arithmetic is poor.

Question 6 (*by 4:00*)

This question was different from the other five in that the child had to think back to know when to get started to arrive in Atlanta by 4:00. This question was intended

to be relatively easy, with only one end of the interval involving nonzero minutes. As can be seen in Table 4, the percentages giving the correct answer increased steadily from 28% in second grade to 41%, 52%, and 54%, respectively, in the subsequent grades. These percentages were similar to those for Question 2, in which only one end of the interval involved nonzero minutes. It was easy for most children to think back from 4:00 to 2:00, but hard to decide what to do with the 20 minutes. The most frequently given incorrect answers were 2:00, 2:20, 2:40, and 1:20. We realized, in retrospect, that it was a mistake to include only one question of this kind because analysis is very difficult when there is only one question requiring thinking back.

When a child remained silent for a long time, a pencil and picture of clocks without hands were offered. Almost all children declined these offers, saying that the tools would not help. Among the negligible few who accepted this offer, not a single child was helped by the tools. The two who accepted the offer of a pencil used an "algorithm" similar to the one in Figure 6b. The child who used this "algorithm" clearly knew that there are 60 minutes in an hour, but this isolated bit of knowledge did not help him. The three who accepted the offer of a picture of clocks tried to draw the hands but were unsuccessful. (A digital clock was not offered because the questions the children read had numerals that looked exactly like those on a digital clock.)

Finally, a frequently observed problem was children's poor mental computation. For example, a surprising number of children made errors such as 40 – 15 = 35. The interviews were conducted orally, and the children did not seem to notice such errors.

DISCUSSION

The major finding of the present study is that the difficulty of elapsed time is due mainly to children's inability to coordinate hierarchical units. In this study, the higher order units involved hours, and the lower order units involved minutes. When a question concerned only whole hours, as in Question 1, the percentages giving the correct answer were high (69%, 93%, 97%, and 97%, respectively, in Grades 2–5). However, when a question involved 30 minutes, as in Question 2, the percentages giving the correct answer decreased dramatically to 28%, 45%, 61%, and 60%, respectively, in Grades 2–5. These decreases were statistically significant ($p < .001$) at all grade levels. The most frequently occurring incorrect answer to Question 2 (8:30–11:00) was 3 hours 30 minutes, obtained by counting 3 hours from 8:00 to 11:00 and adding 30 minutes. The large proportion of students who gave this incorrect answer to such an easy question at all the grade levels 2–5 (34%, 38%, 33%, and 31%, respectively) was surprising.

Generally speaking, children's inability to coordinate hours and minutes was demonstrated in two ways. One was their separate treatment of hours and minutes, which was described previously. The other was their treatment of minutes. They sometimes added the minutes when addition made no logical sense, and sometimes subtracted one number from the other when subtraction made no sense. In

Question 4, for example, neither 15 + 10 minutes nor 15 – 10 minutes made sense after saying that more than an hour elapsed between 8:15 and 9:10. Likewise in Question 5, neither 40 + 15 minutes nor 40 – 15 minutes made sense. The only exception to this was Question 3, for which 30 – 15 minutes made logical sense but 15 + 30 minutes did not.

A general point that can be made from Table 5 is that the proportion of children who indiscriminately added the minutes remained relatively constant from Grade 2 to Grade 5, but the proportion using subtraction increased across those grades. As children progress through school, they may come to associate elapsed time with a "difference," which is usually computed by using subtraction. The fifth graders did exceptionally well on Question 3 (69% according to Table 4) and very poorly on Question 4 (40%). These percentages require an explanation. Question 3 was formulated to determine how children dealt with 15 minutes and multiples of 15 minutes. However, it turned out to be very easy for fourth and fifth graders for two reasons: (a) The correct answer could be obtained by dealing with hours and minutes separately, and (b) 15 and 30 are "friendly numbers" that easily combined with older students' growing intuition to use subtraction.

Why did the fifth graders do so poorly (40%) on Question 4 (8:15–9:10)? A possible explanation is that the fifth graders, who had demonstrated a strong tendency (69%) to subtract 15 from 30 in Question 3, jumped at the possibility of subtracting 10 from 15. According to Table 5, 34% of the fifth graders calculated 15–10. If these fifth graders had thought more carefully about the beginning and end points of the interval, they may have noticed that the duration was less than 1 hour.

On the positive side, a major result from this study was that 17% of the second graders invented a sequential way of coordinating hours and minutes in response to Question 5 (6:40 to 9:15)—the most difficult of the six questions. These second graders were not taught how to determine elapsed time because their teachers were convinced of the futility of such instruction. The second graders' invention can thus be said to be truly a construction from within. As can be seen in Figure 7a, these second graders said that there are 20 minutes from 6:40 to 7:00, 2 hours from 7:00 to 9:00, and 15 minutes after 9:00, making a total of 2 hours and 35 minutes.

According to Table 4, similar percentages of second graders also answered Questions 2, 3, 4, and 6 correctly without any instruction—28%, 21%, 21%, and 28%, respectively. Those who answered Question 5 correctly were generally the same children who answered Questions 2, 3, 4, and 6 correctly.

These inventions by approximately 20% of the second graders highlight the role of reflective (constructive) abstraction in the coordination of hours and minutes. As stated earlier, all logico-mathematical relationships are mental relationships that children construct from within. Reflective abstraction explains the construction and coordination of hierarchical units with respect not only to time but also to 10s and 1s (Chandler & Kamii, 2009; Steffe, & Cobb, 1988), multiplicative thinking (Clark & Kamii, 1996; Steffe, 1992), fractions (Steffe & Olive, 2010), and spatial units of units (Reynolds & Wheatley, 1996; Wheatley & Reynolds, 1996).

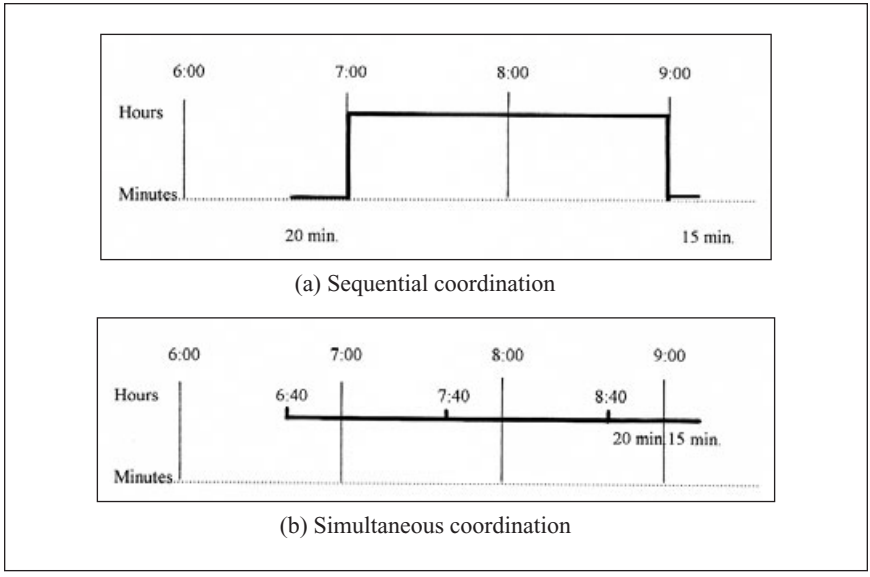


Figure 7. Two ways of coordinating hours and minutes.

In contrast with the sequential way of coordinating hours and minutes, a second way we observed was the simultaneous coordination illustrated in Figure 7b. In this coordination, the student said "7:40" and "8:40" as s/he counted on two fingers, and went on to say "20 minutes and 15 minutes is 35 minutes." This simultaneous counting of hours and minutes appeared for the first time in third grade, but only 39% of the fourth graders and 18% of the fifth graders used this simultaneous method at least once during the interview. One of the third-grade teachers remarked that she never tried to teach this simultaneous method and added that those who doggedly practiced this method were the cognitively more advanced and more creative students. Generally speaking, most of the older students seemed to prefer sequential coordination, but this impression must be verified systematically by future research.

Educational Implications

Two educational implications can be drawn from our study. These differ from the conceptualization of time conveyed in the recently published *Common Core State Standards* (CCSS) in the United States (Common Core State Standards Initiative, 2010). CCSS requires children to "tell and write time in hours and half-hours using analog and digital clocks" (p. 16) in first grade, to tell time to the nearest 5 minutes in second grade (p. 20), and to tell time to the nearest minute in third grade (p. 24). In third grade, CCSS also requires students to "solve word problems involving addition and subtraction of time intervals in minutes, e.g., by representing the problem on a number line diagram" (p. 24). In fourth grade, children are required

to "use the four operations to solve word problems involving distances, intervals of time, liquid volumes, masses of objects, and money" (p. 31). Temporal relationships thus quickly become part of mathematics in third and fourth grades, without first becoming logico-mathematized qualitatively. In fifth grade, time is no longer mentioned, presumably because it has been mastered.

The logico-mathematical nature of time is thus completely overlooked in CCSS. CCSS does not account for the fact that temporal relationships must first be constructed (created) qualitatively from within, by reflective (constructive) abstraction. It also does not reflect an awareness that a major problem—even as late as fifth grade—is the coordination of hierarchical units (hours and minutes). Furthermore, CCSS refers to representations (such as the hours and minutes on clocks and number-line diagrams) but does not take into account that when children use these representations, they must have concepts (mental relationships) to represent.

The educational implications we draw from our study center around children's construction of logico-mathematical relationships, by reflective (constructive) abstraction. Another way of saying *by reflective* (*constructive*) *abstraction* is to say *by thinking* or *by doing one's own thinking*. This means that we must encourage children (a) to think about time in many situations throughout the day, and (b) to do their own thinking when we try to teach elapsed time more specifically.

Encouraging children to think about time in many situations throughout the day. In school, children are often given directives such as "It is time to (go to lunch)" and "You have 5 minutes (to finish what you are doing)." Children who are thus never encouraged to think about durations cannot be expected to become able to reason about it. It therefore seems desirable to conduct research to test the hypothesis that encouraging them to think about durations throughout the day results in higher percentages of correct answers than those presented in Table 4. To encourage them to think about durations, it seems desirable to ask questions such as the following and to encourage debate among students:

- Do we have enough time to read a story before lunch? (If it is 10:45 now, and the class must leave for lunch at 11:17, this question may stimulate a fruitful discussion.)
- What time does the bus come? How many minutes do we need to get ready for it?
- If each person gets 10 minutes to give a book report, how much time will the class need for all the reports?

Encouraging children to do their own thinking when we try to teach elapsed time. The use of a time line is recommended by the authors of CCSS (Common Core State Standards Initiative, p. 24) as well as of many books and articles. However, authors differ in the advice they give to teachers, and it seems desirable to conduct research to determine which approach produces higher percentages of correct answers than those reported in Table 4.

Van de Walle (2007), for example, recommends children's use of an empty time

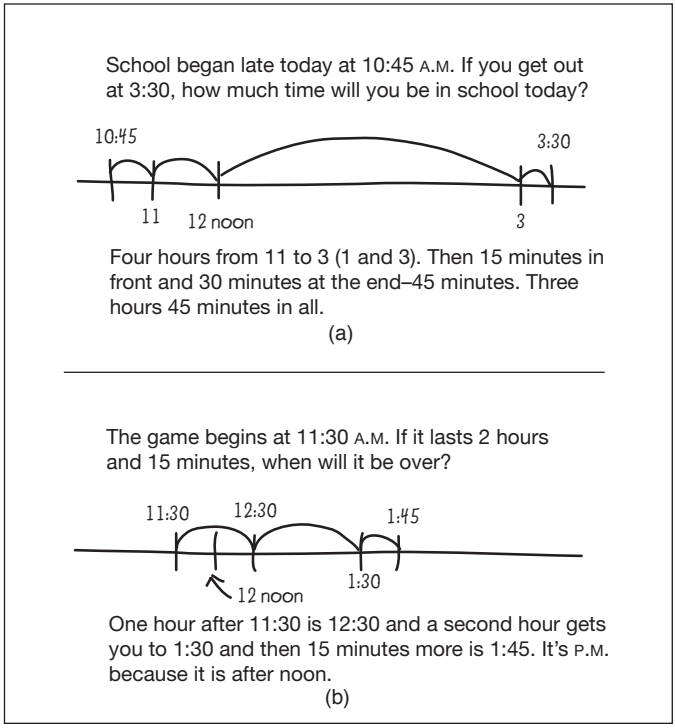


Figure 8. Two ways of using an empty time line from *Elementary and Middle School Mathematics,* 6th ed., by John Van de Walle (2007), p. 392. Copyright 2007 by Pearson Education, Inc.

line that permits them to use it as they wish. He presents the two examples shown in Figure 8 to illustrate two different ways in which an empty time line was used. (The answer of 3 hours 45 minutes in Figure 8a should have been 4 hours 45 minutes.)

By contrast, Moone and de Groot (2005) make many specific recommendations such as drawing large jumps for hours and small jumps for minutes so that time lines will serve as scaffolds toward the conventional algorithm involving regrouping (see Figure 6a). We believe that these specific instructions prevent children from doing their own thinking. In fact, we believe that the algorithm of subtraction may interfere with children's additive way of thinking. As can be seen In Figures 7a and 7b, as well as in Figures 8a and 8b, children think additively. We think that children do not become logical thinkers by being told what to draw and what to write, but this is a hypothesis yet to be tested by pedagogical research.

More research is necessary to answer the specific questions raised by our study. We nevertheless hope that our study will be helpful to teachers who have had no assistance from the research literature about why elapsed time is so difficult to teach.

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